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UNIVERSAL NON-COMPACT OPERATORS BETWEEN SUPER-REFLEXIVE BANACH SPACES AND THE EXISTENCE OF A COMPLEMENTED COPY OF HILBERT SPACE

BY
S. J. DILWORTH
Department of Mathematics, University of Missouri — Columbia, Columbia, MO 65211, USA

ABSTRACT
Suppose that $1 < p \leqq 2$, $2 \leqq q < \infty$. The formal identity operator $I : l_p \to l_q$ factorizes through any given non-compact operator from a p-smooth Banach space into a q-convex Banach space. It follows that if X is a 2-convex space and Y is an infinite dimensional subspace of X which is isomorphic to a Hilbert space, then Y contains an isomorphic copy of l_2 which is complemented in X.

1. Basic sequences and non-compact operators

The existence of a basic sequence which bears a special relation to a given finite collection of non-compact operators is proved in Proposition 1.3 below. This is applied to obtain some results about the existence of quasi-complements and to obtain an extension of a theorem on the existence of a universal non-compact operator; the latter result provides the motivation for the theorems described in the abstract, which are proved in Section 2.

Suppose that T is a bounded operator from a Banach space X into a Banach space Y. The quantity $c(T)$ is defined by $c(T) = \inf\{\| T|_M \| : M$ is a closed subspace of finite codimension in $X\}$. It is proved in [7] and [15] that T is a compact operator if and only if $c(T) = 0$. For completeness a simple proof of this fact will now be given.

PROPOSITION 1.1. *T is compact if and only if $c(T) = 0$.*

PROOF. First suppose that $c(T) > \delta > 0$. We shall construct a sequence $(x_k)_{k=1}^{\infty}$ in the unit ball of X such that $\| Tx_k \| > \delta$ and $\| Tx_j - Tx_k \| > \delta$ for all $k \neq j$.

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Suppose that $x_1,\ldots,x_n$ have been constructed with these properties for all $1 \leqq j < k \leqq n$. Select $f_1,\ldots,f_n$ in Y^* such that $f_i(Tx_i) > \delta$ and $\|f_i\| \leqq 1$. Then $M = \bigcap_{i=1}^n \ker(T^*f_i)$ is of finite codimension in X, and so there exists x_{n+1} in M with $\|x_{n+1}\| \leqq 1$ and $\|Tx_{n+1}\| > \delta$. Moreover, $\|Tx_{n+1} - Tx_i\| \geqq f_i(Tx_i - Tx_{n+1}) > \delta$ $(1 \leqq i \leqq n)$, which proves the next step in the induction. Since the sequence $(Tx_n)_{n=1}^\infty$ has no convergent subsequence, it follows that T is non-compact. To prove the converse assertion we shall assume that $c(T) = 0$ and show that T is then compact. To this end let $(x_n)_{n=1}^\infty$ be any sequence in the unit ball of X and let $\varepsilon > 0$ be given. There exists a closed subspace M of finite codimension with $\|T|_M\| < \varepsilon$; let N be any (finite dimensional) complement of M and let P be the projection which is parallel to M and whose range is N. There exists a subsequence $(x_{n_k})_{k=1}^\infty$ such that $\|Px_{n_j} - Px_{n_k}\| < \varepsilon$ for all $j \neq k$. We have

$$\|(I-P)(x_{n_j} - x_{n_k})\| \leqq \|x_{n_j} - x_{n_k}\| + \|Px_{n_j} - Px_{n_k}\| \leqq 2 + \varepsilon,$$

and so

$$\begin{aligned}\|T(x_{n_j} - x_{n_k})\| &\leqq \|T\| \, \|P(x_{n_j} - x_{n_k})\| + \|T|_M\| \, \|(I-P)(x_{n_j} - x_{n_k})\| \\ &\leqq \|T\|\varepsilon + \varepsilon(2+\varepsilon).\end{aligned}$$

Since ε is arbitrary it follows easily that $(Tx_n)_1^\infty$ has a convergent subsequence, and so T is compact.

LEMMA 1.2. *Suppose that $T_i : X \to Y_i$ $(1 \leqq i \leqq n)$ are non-compact operators. There exists $\delta > 0$ such that for every closed subspace M of finite codimension in X there exists $x \in M$ with $\|x\| \leqq 1$ and $\|T_ix\| > \delta$ $(1 \leqq i \leqq n)$.*

PROOF. There exists $\eta > 0$ such that $c(T_i) > \eta$ $(1 \leqq i \leqq n)$. We shall show by induction that we may take $\delta = \eta/2^n$. Suppose the result is true for $n-1$. There exists $y \in M$ with $\|y\| \leqq 1$ and $\|T_iy\| > \eta/2^{n-1}$ $(1 \leqq i \leqq n-1)$. If $\|T_ny\| > \eta/2^n$, then we are done. The other possibility is that $\|T_ny\| \leqq \eta/2^n$. In this case we choose norm one functionals $(f_i)_{i=1}^{n-1}$ such that $f_i(T_iy) > \eta/2^{n-1}$. Since $L = M \cap \bigcap_{i=1}^{n-1} \ker(T_i^*f_i)$ is of finite codimension there exists $z \in L$ such that $\|z\| \leqq 1$ and $\|T_nz\| > \eta$. Let $x = \frac{1}{2}(y+z)$ and observe that $x \in M$ and $\|x\| \leqq 1$. For $1 \leqq i \leqq n-1$, we have

$$\|T_ix\| \geqq \tfrac{1}{2} f_i(T_iy + T_iz) > \eta/2^n.$$

Moreover,

$$\|T_nx\| \geqq \tfrac{1}{2}\|T_nz\| - \tfrac{1}{2}\|T_ny\| > (\tfrac{1}{2} - (\tfrac{1}{2})^n)\eta \geqq (\tfrac{1}{2})^n\eta,$$

provided $n \geqq 2$. This completes the induction.

PROPOSITION 1.3. *Suppose that $T_i : X \to Y_i$ are non-compact operators. There exists $\delta > 0$ and a normalized basic sequence $(x_k)_{k=1}^{\infty}$ in X such that $\| T_i x_k \| > \delta$ $(1 \leqq i \leqq n,\ k \geqq 1)$ and $(T_i x_k)_{k=1}^{\infty}$ is a basic sequence in Y_i $(1 \leqq i \leqq n)$. Moreover, given $K > 1$, we may ensure that the basis constant of each of these basic sequences is at most K.*

PROOF. Let $\delta > 0$ be as given by the statement of Lemma 1.2. The proof is an adaptation of Mazur's argument for the construction of a basic sequence in an infinite dimensional Banach space (see e.g. [10, p. 4]). Let $(\varepsilon_j)_{j=1}^{\infty}$ be a sequence of positive numbers satisfying $\Pi_1^{\infty}(1+\varepsilon_j) = K < \infty$. Suppose that $x_1, \dots, x_k$ have been obtained such that for all choices of scalars $\lambda_1, \dots, \lambda_k$ the following statements are true:

(a) $\| \Sigma_{j=1}^{r} \lambda_j x_j \| \leqq \Pi_1^{s}(1+\varepsilon_j) \| \Sigma_{j=1}^{s} \lambda_j x_j \|$ $(1 \leqq r \leqq s \leqq k)$;

(b) $\| \Sigma_{j=1}^{r} \lambda_j T_i x_j \| \leqq \Pi_1^{s}(1+\varepsilon_j) \| \Sigma_{j=1}^{s} \lambda_j T_i x_j \|$ $(1 \leqq r \leqq s \leqq k,\ 1 \leqq i \leqq n)$.

Let E and F_i $(1 \leqq i \leqq n)$ denote the subspaces spanned by $x_1, \dots, x_k$ and $T_i x_1, \dots, T_i x_k$ respectively, and let $w_1, \dots, w_p$ and $z_1^i, \dots, z_{q_i}^i$ be $(\frac{1}{2})\varepsilon_{k+1}$-nets of the unit spheres of E and F_i respectively. Now select $f_r \in X^*$ $(1 \leqq r \leqq p)$ and $g_r^i \in Y_i^*$ $(1 \leqq r \leqq q_i,\ 1 \leqq i \leqq n)$ such that $f_r(w_r) = \| f_r \| = 1$ and $g_r^i(z_r^i) = \| g_r^i \| = 1$; let

$$M = \{ x \in X : f_r(x) = T_i^* g_s^i(x) = 0 : 1 \leqq r \leqq p, 1 \leqq i \leqq n, 1 \leqq s \leqq q_i \}.$$

Then M is of finite codimension and so by our choice of δ there exists x_{k+1} in M with $\| x_{k+1} \| = 1$ and $\| T_i x_{k+1} \| > \delta$ $(1 \leqq i \leqq n)$. Suppose that y lies on the unit sphere of E; there exists a w_r such that for each scalar λ, we have

$$\begin{aligned} \| y + \lambda x_{k+1} \| &\geqq \| w_r + \lambda x_{k+1} \| - \tfrac{1}{2}\varepsilon_{k+1} \\ &\geqq f_r(w_r + \lambda x_{k+1}) - \tfrac{1}{2}\varepsilon_{k+1} \\ &\geqq 1 - \tfrac{1}{2}\varepsilon_{k+1} \\ &\geqq 1/(1+\varepsilon_{k+1}). \end{aligned}$$

It follows that (a) holds with k replaced by $k+1$, and a similar verification shows that (b) holds with k replaced by $k+1$, which completes the induction. The estimate for the basis constants is immediate.

COROLLARY 1.4. (a) *Suppose that $T_i : X \to Y_i$ are non-compact operators $(1 \leqq i \leqq n)$. There exists an infinite dimensional closed subspace N of X such that the restriction of each T_i to N is non-compact and injective.*

(b) *Suppose that $X_1, \dots, X_n$ are closed subspaces of infinite codimension in the*

Banach space X. There exists a closed subspace N of X such that the restriction to N of the quotient map $X \to X/X_i$ is non-compact and injective for each $1 \leqq i \leqq n$.

PROOF. (a) Let $(x_k)_{k=1}^{\infty}$ be a basic sequence whose existence is asserted in the statement of Proposition 1.3, and let N be the closed linear span of this sequence. The result follows at once from the properties of this sequence which are stated in Proposition 1.3.

(b) This follows from part (a) by taking T_i to be the quotient mapping $X \to X/X_i$.

Let M and N be closed subspaces of a Banach space X. We recall that N is said to be a quasi-complement of M if $M \cap N = 0$ and $M + N$ is dense in X. It is proved in [11] that every subspace of a separable Banach space possesses a quasi-complement, while it is proved in [9] that if Γ is uncountable then $c_0(\Gamma)$ does not possess a quasi-complement in $l_\infty(\Gamma)$. Now suppose that X is a separable infinite dimensional Banach space which does not contain l_1 and that $T: l_1 \to X$ is a quotient mapping. Since any infinite dimensional subspace of l_1 contains l_1 it must follow that if $\ker T \subset N \subset l_1$, where N is a closed subspace, and that if $\ker T$ is complemented in N, then $N/\ker T$ is finite dimensional. The above remarks are relevant to the following proposition, which translates Corollary 1.4 (b) into a statement about quasi-complements.

COROLLARY 1.5. *Suppose that $X_1, \ldots, X_n$ are closed subspaces of infinite codimension in the Banach space X. There exists a subspace N of X such that N is a quasi-complement of X_i in the closure of $N + X_i$ such that the restriction to N of the quotient map $X \to X/X_i$ is non-compact for each $1 \leqq i \leqq n$.*

It was proved in [5] that the formal identity operator $I: l_1 \to l_\infty$ is a universal non-compact operator in the sense that it can be factorized through any given non-compact operator. The next proposition shows that it is possible to obtain simultaneous factorizations through any finite collection of non-compact operators.

COROLLARY 1.6. *Suppose that $T_i: X \to Y_i$ $(1 \leqq i \leqq n)$ are non-compact operators. There exist operators $U: l_1 \to X$ and $V_i: Y_i \to l_\infty$ $(1 \leqq i \leqq n)$ together with the following factorizations of the formal identity $I: l_1 \to l_\infty$:*

$$l_1 \xrightarrow{U} X \xrightarrow{T_i} Y_i \xrightarrow{V_i} l_\infty \qquad (1 \leqq i \leqq n).$$

Moreover, given $K > 1$, we may ensure that $\|U\| \leqq 1$ and $\|V_i\| \leqq 2K/\delta$ $(1 \leqq i \leqq n)$, where $\delta > 0$ is as defined in Proposition 1.3.

PROOF. Let $(x_k)_{k=1}^{\infty}$ be a basic sequence whose existence is asserted in the statement of Proposition 1.3; let U be defined by $U(e_k) = x_k$, where $(e_k)_{k=1}^{\infty}$ is the unit vector basis of l_p $(1 \leqq p \leqq \infty)$, with extension to l_1 by linearity. Since $(T_i x_k)_{k=1}^{\infty}$ is a basic sequence for each $1 \leqq i \leqq n$, it follows that the mapping $T_i x_k \to e_k$ $(k \geqq 1)$ extends to a bounded operator from the closed linear span of $(T_i x_k)_{k=1}^{\infty}$ into l_∞; but l_∞ is a $\mathscr{P}_1$ space and so the latter operator has a bounded extension $V_i : Y_i \to l_\infty$. The estimates for $\| U \|$ and $\| V_i \|$ are easily verified.

We now prove a dual version of the previous result. It should be noted that estimates could easily be given for the norms of the operators appearing in the statement.

COROLLARY 1.7. *Suppose that $T_i : X_i \to Y$ are non-compact operators $(1 \leqq i \leqq n)$. There exist operators $U_i : l_1 \to X_i$ and $V : Y \to l_\infty$ together with the following factorizations of the formal identity $I : l_1 \to l_\infty$:*

$$l_1 \xrightarrow{U_i} X_i \xrightarrow{T_i} Y \xrightarrow{V} l_\infty \qquad (1 \leqq i \leqq n).$$

PROOF. We shall sketch the proof only in the case $n = 2$ as the extension to the general case is then a matter of routine. The argument of Proposition 1.3 can be modified to show that there exist $\delta > 0$ and normalized sequences $(x_k)_{k=1}^{\infty}$ in X_1 and $(y_k)_{k=1}^{\infty}$ in X_2 such that $(z_k)_{k=1}^{\infty}$, which is defined by $z_{2k} = T_1 x_k$ and $z_{2k+1} = T_2 y_k$ $(k \geqq 1)$, is a basic sequence in Y with $\| z_k \| > \delta$. We define $U_1 : l_1 \to X_1$ and $U_2 : l_1 \to X_2$ by $U_1(e_k) = x_k$ and $U_2(e_k) = y_k$ with extension to l_1 by linearity. It follows easily from the fact that $(z_k)_{k=1}^{\infty}$ is basic that the mapping given by $z_{2k} \to e_k$ and $z_{2k-1} \to e_k$ $(k \geqq 1)$ extends to a bounded operator from the closed linear span of $(z_k)_{k=1}^{\infty}$ into l_∞. Since l_∞ is a $\mathscr{P}_1$ space the latter operator has a bounded extension $V : Y \to l_\infty$, and this completes the proof.

We end this section with the remark that the case $n = 1$ of Proposition 1.3 will follow very simply from a result of A. Pełczyński ([13]) to the effect that every bounded sequence which has no weakly convergent subsequence contains a basic subsequence. The latter result gives rise in a straightforward manner to the following analogue of Corollary 1.4 and Corollary 1.5 for non-weakly compact operators in the case $n = 1$.

PROPOSITION 1.8. (a) *Suppose that $T : X \to Y$ is a non-weakly compact operator. There exists an infinite dimensional closed subspace N of X such that the restriction of T to N is non-weakly compact and injective.*

(b) *Suppose that X_1 is a closed reflexive subspace of the non-reflexive Banach* *space X. There exists a subspace N of X such that N is a quasi-complement of X_1 in the closure of $N + X_1$ and such that that the restriction to N of the quotient map $X \to X/X_1$ is non-weakly compact.*

It is unknown to me, however, whether the non-weakly compact analogue of Corollary 1.4 is true for general values of n.

2. Universal non-compact operators between super-reflexive spaces

The modulus of convexity δ_X of the Banach space X is defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|\tfrac{1}{2}(x+y)\| : \|x\| \leqq 1, \|y\| \leqq 1, \|x-y\| \geqq \varepsilon\}$$

for $0 \leqq \varepsilon \leqq 2$; its modulus of smoothness ρ_X is defined by

$$\rho_X(t) = \sup\{\tfrac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 : \|x\| = \|y\| = 1\}$$

for $0 \leqq t \leqq \infty$.

X is said to be q-convex $(2 \leqq q < \infty)$ if there exist a constant $C > 0$ and an equivalent norm on X for which the modulus of convexity δ satisfies $\delta(\varepsilon) \geqq C\varepsilon^q$ $(0 \leqq \varepsilon \leqq 2)$. X is said to be p-smooth $(1 < p \leqq 2)$ if there exist a constant C and an equivalent norm on X for which the modulus of smoothness ρ satisfies $\rho(t) \leqq Ct^p$. X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for $\varepsilon > 0$. The moduli of convexity and smoothness of a Banach space and its dual are related by the following duality formulae of J. Lindenstrauss ([8]):

$$\rho_X(t) = \sup_{0<\varepsilon\leqq 2}\{t\varepsilon/2 - \delta_{X^*}(\varepsilon)\} \qquad (t > 0);$$

$$\delta_{X^*}(\varepsilon) \geqq \sup_{t>0}\{t\varepsilon/2 - \rho_X(t)\} \qquad (0 < \varepsilon \leqq 2).$$

These duality formulae have the consequence, which is used below, that X is q-convex if and only if X^* is p-smooth $(1/p + 1/q = 1)$. Finally, let us recall a deep theorem of G. Pisier ([16]) to the effect that every so-called super-reflexive Banach space (see [4] for some characterizations of super-reflexivity) is both q-convex and p-smooth for some q and p. It was proved earlier by P. Enflo ([3]) that every super-reflexive Banach space admits a uniformly convex norm.

The modulus of convexity of the sequence space l_r is of power type $q = \max(2, r)$ and its modulus of smoothness is of power type $p = \min(2, r)$. Moreover, l_r is neither q'-convex nor p'-smooth for any $q' < q$ or $p' > p$. Suppose now that $1 < p \leqq 2$ and that $2 \leqq q < \infty$; then the formal identity operator $I : l_p \to l_q$ is bounded and non-compact. The purpose of this article is to prove the following theorem, which asserts that the latter operator is a universal non-compact operator from a p-smooth Banach space into a q-convex space.

THEOREM 2.1. *Suppose that X is p-smooth, that Y is q-convex, and that $T: X \to Y$ is non-compact. There exist operators $U: l_p \to X$ and $V: Y \to l_q$ together with the following factorization of the formal identity $I: l_p \to l_q$:*

$$l_p \xrightarrow{U} X \xrightarrow{T} Y \xrightarrow{V} l_q.$$

It should be noted that in this theorem and in the other factorization theorems of this section it would be possible to give estimates for the norms of the relevant operators. The proof of Theorem 2.1 will follow from some facts about basic sequences in uniformly convex spaces. We need to introduce some notation for the purpose of stating the next two propositions. Let $(x_k)_{k=1}^{\infty}$ be a basic sequence in the Banach space X, and let P_n be the natural projection associated with this sequence which is defined on the closed linear span of $(x_k)_{k=1}^{\infty}$ and whose range is the linear span of $x_1, \ldots, x_n$. Let $K_n = \| P_n \|$ and let $S_n = \sum_{k=1}^{n} x_k$ $(n \geqq 1)$. The proof of the following proposition is adapted from a similar result for monotone basic sequences which was proved in [16].

PROPOSITION 2.2. *Suppose that* $\sup_{n \geqq 1} K_n \| S_{n+1} \| \leqq 1$. *Then*

$$\| x_1 \| + \sum_{n=2}^{\infty} \delta_X(\| x_n \|) \leqq \prod_{1}^{\infty} K_j + \sum_{j=1}^{\infty} (1 - 1/K_j).$$

PROOF. Since $\| S_n \| \leqq K_n \| S_{n+1} \|$, we have

$$\frac{1}{2} \left\| \frac{S_n + S_{n+1}}{K_n \| S_{n+1} \|} \right\| \leqq 1 - \delta_X \left(\frac{\| x_{n+1} \|}{K_n \| S_{n+1} \|} \right);$$

i.e.

$$\| S_n + \tfrac{1}{2} x_{n+1} \| + K_n \| S_{n+1} \| \delta_X \left(\frac{\| x_{n+1} \|}{K_n \| S_{n+1} \|} \right) \leqq K_n \| S_{n+1} \|.$$

Because $K_n \| S_{n+1} \| \leqq 1$ and because $\varepsilon \to \delta_x(\varepsilon)/\varepsilon$ is increasing on $[0,2)$ (see [16]) we obtain

$$\delta_X(\| x_{n+1} \|) \leqq K_n \| S_{n+1} \| \, \delta_X \left(\frac{\| x_{n+1} \|}{K_n \| S_{n+1} \|} \right).$$

Combining the latter with the relation $\| S_n \| \leqq K_n \| S_n + \frac{1}{2} x_{n+1} \|$ gives

$$\frac{1}{K_n} \| S_n \| + \delta_X(\| x_{n+1} \|) \leqq K_n \| S_{n+1} \|,$$

and so

$$\delta_X(\|x_{n+1}\|) \leqq (K_n\|S_{n+1}\| - \|S_n\|) + \|S_n\|(1 - 1/K_n)$$
$$\leqq \left(\left(\prod_1^n K_j\right)\|S_{n+1}\| - \left(\prod_1^{n-1} K_j\right)\|S_n\|\right) + \|S_n\|(1 - 1/K_n).$$

Summing from $n = 1$ gives

$$\sum_{n=2}^{\infty} \delta_X(\|x_n\|) \leqq \left(\prod_1^{\infty} K_j\right)\sup_{n \geqq 1}\|S_n\| - \|x_1\| + \left(\sum_{j=1}^{\infty}(1 - 1/K_j)\right)\sup_{n \geqq 1}\|S_n\|,$$

and the result follows.

The Lindenstrauss duality formulae can be used to prove the following dual analogue of Proposition 2.2, whose proof is omitted.

PROPOSITION 2.3. *Suppose that* $\Pi_1^{\infty} K_j < \infty$ *and that* $\Sigma_{n=1}^{\infty} \rho_X(\|x_n\|) \leqq 1$. *There exists a constant* C *(depending only on the* K_j*'s) such that* $\sup_n \|S_n\| \leqq C$.

We shall now suppose that $(x_n)_{n=1}^{\infty}$ is a normalized basic sequence, that $1 < p \leqq 2$, and that $2 \leqq q < \infty$. This sequence will be said to satisfy an upper p-estimate if there exists a constant A such that $\|\Sigma_1^{\infty} \lambda_n x_n\| \leqq A(\Sigma_1^{\infty}|\lambda_n|^p)^{1/p}$ for all choices of scalars $\lambda_1, \lambda_2, \ldots$, and will be said to satisfy a lower q-estimate if there exist a constant $A > 0$ such that $\|\Sigma_1^{\infty} \lambda_n x_n\| \geqq A(\Sigma_1^{\infty}|\lambda_n|^q)^{1/q}$ for all choices of scalars. Finally, $(x_n)_{n=1}^{\infty}$ will be said to be a sufficiently monotone basic sequence if (in the notation above) $\Pi_1^{\infty} K_n < \infty$. The following proposition is an immediate consequence of the last two results.

PROPOSITION 2.4. *Suppose that* $(x_n)_{n=1}^{\infty}$ *is a sufficiently monotone normalized basic sequence in the Banach space* X.

(a) *If* X *is* p*-smooth then* $(x_n)_{n=1}^{\infty}$ *satisfies an upper* p*-estimate.*

(b) *If* X *is* q*-convex then* $(x_n)_{n=1}^{\infty}$ *satisfies a lower* q*-estimate.*

It is indeed fortunate that there is an abundant supply of sufficiently monotone basic sequences, and the following proposition records this fact. The proof is implicit in the well-known Mazur technique for extracting a basic subsequence from a weakly null sequence (se e.g. [2, p. 42]).

PROPOSITION 2.5. *Suppose that* $(x_n)_{n=1}^{\infty}$ *is a weakly null normalized sequence in a Banach space. Then* $(x_n)_{n=1}^{\infty}$ *contains a sufficiently monotone basic subsequence.*

PROOF OF THEOREM 2.1. By Proposition 1.3 there exists $\delta > 0$ and a normalized basic sequence $(x_n)_{n=1}^{\infty}$ such that $(Tx_n)_{n=1}^{\infty}$ is a basic sequence with $\|Tx_n\| > \delta$.

Since X is reflexive it follows that $(x_n)_{n=1}^{\infty}$ is weakly null, and so by Proposition 2.5 we may assume that $(x_n)_{n=1}^{\infty}$ is sufficiently monotone (after extracting a suitable subsequence and relabelling). By Proposition 2.4 we may further assume that $(x_n)_{n=1}^{\infty}$ satisfies an upper p-estimate. Hence we obtain the following factorization of the formal identity $I : l_p \to l_\infty$ just as in the proof of Corollary 1.6:

$$l_p \xrightarrow{W} X \xrightarrow{T} Y \xrightarrow{S} l_\infty .$$

Dualizing, we obtain the following factorization of the formal identity $I : l_1 \to l_{p'}$:

$$l_1 \xrightarrow{\iota} l_\infty^* \xrightarrow{S^*} Y^* \xrightarrow{T^*} X^* \xrightarrow{W^*} l_{p'},$$

where $1/p + 1/p' = 1$ and $\iota : l_1 \to l_\infty^*$ is the natural inclusion of l_1 in its second dual. Let $(e_n)_{n=1}^{\infty}$ denote the unit vector basis of l_p $(1 \leqq p \leqq \infty)$. Since Y^* is reflexive it follows that $(S^*\iota(e_n))_{n=1}^{\infty}$ admits a weakly convergent subsequence. Moreover, $W^*T^*S^*\iota(e_n) = e_n$ and so $(S^*\iota(e_n))_{n=1}^{\infty}$ contains no norm convergent subsequence. Hence there exists x in Y^* such that $(S^*(e_n) - x)_{n=1}^{\infty}$ contains a weakly null subsequence which is bounded away from zero. It follows from the Lindenstrauss duality formulae that Y^* is q'-smooth, where $1/q + 1/q' = 1$. By Propositions 2.4 and 2.5 there exists a subsequence $(e_{n_k})_{k=1}^{\infty}$ such that $(S^*\iota(e_{n_k}) - x)_{k=1}^{\infty}$ satisfies an upper q'-estimate. Let $y_k = S^*\iota(e_{n_k}) - x$ and let $\psi : l_{q'} \to Y^*$ be defined by $\psi(e_k) = y_k$ $(k \geqq 1)$ with extension by linearity. Now $W^*T^*(y_k) = e_{n_k} - W^*T^*(x)$; since $(y_k)_{k=1}^{\infty}$ is weakly null it follows that $W^*T^*(x) = 0$. Let the operator $\phi : l_{p'} \to l_{p'}$ be defined by $\phi(e_j) = 0$ for all $j \not\in \{n_k : k \geqq 1\}$ and $\phi(e_{n_k}) = e_k$ for all $k \geqq 1$. It is now readily verified that $\phi W^*T^*\psi$ is the formal identity from l'_q into l'_p. The statement of the theorem follows by dualizing once more.

The proof of Theorem 2.1 contains the proof of the following proposition.

Proposition 2.6. *Suppose that $T : X \to Y$ is a non-compact operator.*

(a) *If X is a p-smooth Banach space then the formal identity $I : l_p \to l_\infty$ admits the following factorization:*

$$l_p \xrightarrow{U} X \xrightarrow{T} Y \xrightarrow{V} l_\infty .$$

(b) *If Y is a q-convex Banach space then the formal identity $I : l_1 \to l_q$ admits the following factorization:*

$$l_1 \xrightarrow{U} X^{**} \xrightarrow{T^{**}} Y \xrightarrow{V} l_q .$$

Using the full force of Proposition 1.3 one can prove versions of Theorem 2.1 and Proposition 2.6 which guarantee the existence of simultaneous factorizations through finite collections of non-compact operators: we state without proof one such result.

PROPOSITION 2.7. *Suppose that X is p-smooth, that $Y_1,\dots,Y_n$ are q-convex, and that $T_i : X \to Y_i$ are non-compact operators* ($1 \leqq i \leqq n$). *There exist operators $U : l_p \to X$ and $V_i : Y_i \to l_q$ together with the following factorizations of the formal identity $I : l_p \to l_q$*:

$$l_p \xrightarrow{U} X \xrightarrow{T_i} Y_i \xrightarrow{V_i} l_q.$$

For the remainder of this article the term Hilbert space will be used to refer only to an infinite dimensional Hilbert space, and the results will be invalid without this provision. Of special interest is the case $p = q = 2$ in Theorem 2.1, which has the following consequence.

COROLLARY 2.8. *Suppose that X is 2-smooth, that Y is 2-convex, and that $T : X \to Y$ is non-compact. Then the range of T contains an isomorphic copy of Hilbert space which is complemented in Y.*

PROOF. Theorem 2.1 asserts the existence of the following factorization of the identity operator on l_2:

$$l_2 \xrightarrow{U} X \xrightarrow{T} Y \xrightarrow{V} l_2.$$

It is now easily seen that UVT is a projection on X whose range is the range of U and that TUV is a projection on Y whose range is the range of TU. Moreover, the ranges of both these projections are isomorphic to l_2.

The next theorem is simply an important special case of the previous corollary.

THEOREM 2.9. *Suppose that X is a 2-convex Banach space and that Y is a subspace of X which is isomorphic to a Hilbert space. Then Y contains an isomorphic copy of l_2 which is complemented in X.*

PROOF. There exists a Hilbert space H and an isomorphic embedding $H \xrightarrow{U} Y \xrightarrow{\iota} X$, where ι is the inclusion operator. Since H is 2-smooth the result follows from Corollary 2.8.

PROPOSITION 2.10. *Suppose that X is a 2-convex Banach space. The following are equivalent*:

(i) *every operator from a type 2 Banach space into X is compact;*
(ii) *every operator from a 2-smooth Banach space into X is compact;*
(iii) *every operator from a Hilbert space into X is compact;*
(iv) *X does not contain a subspace isomorphic to Hilbert space;*
(v) *X does not contain a complemented subspace isomorphic to Hilbert space.*

PROOF. Corollary 2.8 and Theorem 2.9 imply the equivalence of (ii), (iii), (iv) and (v); (i) implies (ii) is evident. To prove that (iii) implies (i), suppose that Y is of type 2 and that $T: Y \to X$ is non-compact. Since X is of cotype 2 it follows from a theorem of S. Kwapien (see e.g. [17, Theoreme 1.2]) that T admits the following factorization through a Hilbert space H:

$$Y \xrightarrow{U} H \xrightarrow{V} X;$$

clearly V is non-compact, and so (iii) implies (i).

Let us recall that an operator $T: X \to Y$ is said to be strictly singular if its restriction to any closed infinite dimensional subspace of X fails to be an isomorphism; T is said to be strictly co-singular if its composition with any infinite rank quotient operator on Y fails to be a quotient operator on X. Suppose that (Ω, Σ, μ) is any measure space with an infinite σ-field Σ, that $2 \leqq q < \infty$, and that $2 < p < \infty$. There is a projection P on $L_q(\mu)$ whose range E is isomorphic to l_2 and there is a subspace F of $L_p(\mu)$ which is isomorphic to l_p. The following composition, in which ϕ and ψ are isomorphisms and $I: l_2 \to l_p$ is the formal identity operator, gives a non-compact operator from $L_q(\mu)$ into $L_p(\mu)$ which is both strictly singular and strictly cosingular:

$$L_q(\mu) \xrightarrow{P} E \xrightarrow{\phi} l_2 \xrightarrow{I} l_p \xrightarrow{\psi} F \longrightarrow L_p(\mu).$$

Similar operators exist in the range $1 < p < 2$, $1 < q \leqq 2$. The following positive result calls to mind the familiar fact that every operator from l_q into l_p is compact for $q > p$.

PROPOSITION 2.11. *Suppose that $1 < p \leqq 2$, that $2 \leqq q < \infty$, and that $T: L_q(\mu) \to L_p(\mu)$ is non-compact. Then T maps a complemented copy of Hilbert space in $L_q(\mu)$ onto a complemented copy of Hilbert space in $L_p(\mu)$. In particular, every strictly singular or strictly co-singular operator from $L_q(\mu)$ into $L_p(\mu)$ is compact.*

PROOF. $L_r(\mu)$ is $\min(r,2)$-smooth and $\max(r,2)$-convex; so the result follows at once from the proof of Corollary 2.8.

I should like to conclude with some remarks about Theorem 2.9. H. P. Rosenthal and A. Pełczyński proved the result of Theorem 2.9 ([14]) in the case $X = L_p(0,1)$ $(1 < p \leqq 2)$; they remarked that their proof would work for any Banach space X which is of cotype 2 and which has an unconditional basis. I would further remark that their proof works for any space X of cotype 2 which has an unconditional finite dimensional decomposition. Since any subspace of a quotient space of a 2-convex space is itself 2-convex, which is a consequence of the Lindenstrauss duality formulae, it follows that the result of Theorem 2.9 is valid for any subspace of a quotient space of $L_p(\mu)$ $(1 < p \leqq 2)$. It is relevant to recall at this point the fact (see [1]) that $L_p(0,1)$ contains uncomplemented isomorphic copies of Hilbert space for each $1 < p < 2$; also relevant is the consequence of Maurey's extension theorem ([12]) to the effect that any isomorphic copy of Hilbert space in a Banach space of type 2 (and a fortiori in a 2-smooth space) is automatically complemented.

The example of $L_1(0,1)$, which is of cotype 2 and possesses subspaces that are isomorphic to a Hilbert space, but none that are complemented, shows that Theorem 2.9 does not extend to encompass all spaces of cotype 2. Moreover, the Kalton–Peck space Z_2, which contains a subspace isomorphic to a Hilbert space which has the property that the quotient of Z_2 by that subspace is also isomorphic to a Hilbert space, is super-reflexive but does not contain any complemented isomorphic copy of Hilbert space (see [6]).

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